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Stabilization and Stability of Nonlinear Dynamical Systems
With Application to Problems of Rigid Bodies Mechanics

Abstract of the doctoral thesis

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Structure of the thesis. The thesis is written in Russian and consists of introduction, five chapters, conclusions, and one appendix. There are five figures scattered in the text. The total volume is 155 pages, the bibliography is comprised of 105 references.

Chapter 1 is a survey of important results on stabilization of controllable systems governed by nonlinear ordinary differential equations, stability theory with respect to a part of the variables, and stabilization of a rigid body by means of jet torques. A brief analysis of certain publications on mathematical modeling of flexible objects by means of rigid bodies systems having elastic hinges is carried out as well. Some open problems are stated therein.

The methodology used is set out in **Chapter 2.** In section 2.1, the problem of global controllability of nonlinear system is treated by the method of oriented manifolds proposed by A.M. Kovalev[1]. The basic concepts of A.F. Filippov's theory of discontinuous differential equations[2] are considered in section 2.2. The next section is focused on application of the reduction principle for critical cases of stability. Stability criteria by G.V. Kamenkov and A.M. Molchanov are used there. The notions of partial stability and partial boundedness of solutions are stated in section 2.4. In this section, some results on partial asymptotic stability[3] obtained by the direct Lyapunov method are formulated.

In **Chapter 3**, we consider the following nonlinear control system:

$$\dot{x} = f(x,u), \tag{1}$$

where $x \in D \subseteq R^n$ is the state, $u \in U \subseteq R^m$ is the control. It is assumed that $f \in C^1(D \times U)$, $0 \in \operatorname{int} D$, $0 \in U$, $f(0,0) = 0$. Let us denote an ε-neighborhood of $x \in D$ as $B(x,\varepsilon)$.

Necessary and sufficient conditions for local reachability and local controllability of (1) are given in section 3.1. These conditions extend the approach by A.M. Kovalev[1] for the local case.

By means of the above results, the following theorem has been proved in section 3.2.

Theorem 1. *Let U be a compact, and let for some $\varepsilon>0$ the system (1) be locally reachable at any point of $B(0,\varepsilon)\setminus\{0\}$. Then there exists a (generally discontinuous) feedback control $u: B(0,\varepsilon) \to U$, $u(0) = 0$ ensuring Lyapunov stability of the closed-loop system:*

$$\dot{x} = f(x,u(x)). \tag{2}$$

Here the solutions of (2) are defined in the sense of Filippov.

Since the notion of reachability is weaker than controllability, Theorem 1 holds for the case when (1) is locally controllable at each point of $B(0,\varepsilon)\setminus\{0\}$. Let us remark that the stability property in this theorem may not be asymptotic.

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[1] Kovalev A.M. (1995*): Controllability criteria and sufficient conditions for dynamical systems to be stabilizable*. Journal of Applied Mathematics and Mechanics, Vol.59, No.3, 379-386.

[2] Filippov A.F. (1988): *Differential equations with discontinuous righthand sides*. Kluwer, Dordrecht.

[3] Rumyantsev V.V., Oziraner A.S. (1987*): Stability and stabilization of motion with respect to a part of the variables*. Nauka, Moscow.

It should be emphasized that the above theorem has some relation with the result by F.H. Clarke, Yu.S. Ledyaev, E.D. Sontag, A.I. Subbotin[1]. The authors of that paper have proved that for any asymptotically null-controllable system there exists a discontinuous stabilizing feedback law. However, their way of defining the feedback solutions ("π trajectories") is not equivalent to Filippov's approach.

In order to show the necessity of applying discontinuous feedback controls, the following example has been considered in the thesis:

$$\dot{x} = f(x,u) \equiv \begin{cases} x & , |u| \leq 1; \\ x + e^{1/(1-u)} & , u > 1; \\ x - e^{1/(1+u)} & , u < -1, \end{cases} \tag{3}$$

where $x \in D = (-e^{-1}, e^{-1})$, $u \in U = [-2, 2]$, $f \in C^{\infty}(D \times U)$. It has been shown that the system (3) is locally controllable at any point of D, however, there is no continuous feedback control ensuring Lyapunov stability of the equilibrium x=0.

On the other hand, Ryan's result[2] implies that, under the conditions of Theorem 1, one cannot guarantee the existence of a feedback law $u(x)$ giving uniform asymptotic stability of the equilibrium point. It means that the assertion of Theorem 1 cannot be strengthen in the general case.

The set of discontinuity points of a stabilizing feedback control is investigated in section 3.3. The main result of such investigation can be formulated as follows.

Theorem 2. *Suppose that U is a convex compact, and that for some ε>0 the system (1) is locally reachable at any point of $B\,(0,\varepsilon)\backslash\{0\}$. Let us assume furthermore that $f(x, u)$ is affine with respect to u and analytic on x. Then there exists a feedback law $u \in C\,(B\,(0,\varepsilon)\backslash M)$, $u(0) = 0$ ensuring Lyapunov stability of the equilibrium x=0, where the dimension of M is at most n-1.*

Section 3.4 is devoted to the problem of optimal stabilization of the system (1) with analytic right-hand side in the critical case when stability is assured by a finite order approximation. Let us assume that for any analytic feedback control $u(x)$ of some class U_0 the corresponding closed-loop system (2) exhibits a critical case of q pairs of purely imaginary roots. By applying the reduction principle[3], stability analysis of (2) leads us to the following so-called "shortened" normalized system:

$$\dot{r}_s = F_s(r_1, r_2, ..., r_q; u), \quad s = 1,2,...,q, \tag{4}$$

where F_s are homogeneous functions of degree $N \geq 3$ with respect to r_1, r_2,..., r_q. The number N characterizes the minimal order of nonlinear terms appearing in the normal form

[1] Clarke F.H., Ledyaev Yu.S., Sontag E.D., Subbotin A.I. (1997): *Asymptotic controllability implies feedback stabilization*. IEEE Trans. on Autom. Control, Vol.42, 1394-1407.

[2] Ryan E.P. (1994): *On Brockett's condition for smooth stabilizability and its necessity in a context of nonsmooth feedback*. SIAM J. Control Optim., Vol.32, No.6, 1597-1604.

[3] Veretennikov V.G. (1984): *Stability and oscillations of nonlinear systems*. Nauka, Moscow (in Russian, Zentralblatt MATH Review Zbl. 0532.34002).

of the shortened system. The right-hand side of (4) depends on parameters, i.e. the Taylor coefficients of a feedback law $u \in U_0$ at $x=0$.

Krasovskii's – Zubov's theorem[1] implies that the necessary and sufficient condition for asymptotic stability of (4) is that the functional

$$\phi^+[r(t)] = \overline{\lim_{t \to +\infty}} \left(t^{1/(N-1)} \left| r(t) \right| \right) \tag{5}$$

should be bounded on the set of solution of (4), where $|r(t)|$ denotes the Euclidean norm of $r(t)$. Let us denote by $r(t; r_0, u)$ the solution of (4) corresponding the feedback control $u \in U_0$ and the initial condition $r(0; r_0, u) = r_0$.

Definition 1. The feedback control $u^* \in U_0$ solves *the problem of optimal stabilization (with respect to asymptotic decreasing of the norm)* if, the corresponding system (4) is asymptotically stable for $u^*(x)$, and for any $u \in U_0$, the following inequality holds:

$$\sup_{r_0 \in R^q} \left\{ \phi^+[r(t; r_0, u^*)] \right\} \le \sup_{r_0 \in R^q} \left\{ \phi^+[r(t; r_0, u)] \right\}. \tag{6}$$

The optimality condition (6) means that the control $u^*(x)$ "uniformly" damps the solutions of (4) in the fastest way as $t \to +\infty$. K. Peiffer and A.Ya. Savchenko[2] have considered a similar problem statement within the framework of passive stabilization of a system with one critical degree of freedom. However, in the above-mentioned paper, the notion of optimality has not been rigorously defined through specifying a value functional on the trajectories.

In subsection 3.4.2, the upper and lower estimates of (5) were obtained for the critical case of q pairs of purely imaginary roots under the condition of Molchanov's stability criterion[3], $N=3$. The proof of these estimates uses an explicit construction of a homogeneous Lyapunov function. A particular case of two pairs of imaginary roots ($q=2$, $N \ge 3$) is considered in subsections 3.4.3, 3.4.4. For this case, by substituting $\rho_s = r_s^2 \ge 0$ into (4), one gets the following system of equations:

$$\dot{\rho}_s = \rho_s \sum_{k_1+k_2=l} G_s^{(k_1,k_2)} \rho_1^{k_1} \rho_2^{k_2}, \quad \rho_s \ge 0, \quad (\text{s=1,2, } l = \left[\frac{N-1}{2}\right]). \tag{7}$$

The main result of section 3.4 is given by Theorem 3.

Theorem 3. *Suppose that the trivial solution of (7) is asymptotically stable in the cone $\rho_1 \ge 0$, $\rho_2 \ge 0$. Then for any solution $r(t)$ of (4) there exists a finite limit:*

$$\phi^+[r(t)] = \lim_{t \to +\infty} \left(t^{1/(N-1)} \left| r(t) \right| \right). \tag{8}$$

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[1] Krasovskii N.N. (1959): *Some problems of stability theory of motion*. Fizmatgiz, Moscow, p. 114-115. (in Russian).
Zubov V.I. (1964): *Methods of A.M. Lyapunov and their application*. P. Noordhoff Ltd., Groningen.
[2] Peiffer K., Savchenko A.Ya. (2000): On the asymptotic behavior of a passively stabilized system with one critical variable. Rend. Acc. Sc. fis. mat. Napoli, Vol.67, 157-168.
[3] Molchanov A.M. (1961): *On the stability in the case of neutral linear approximation*. Doklady AN SSSR, Vol.141(1), 24-27 (in Russian), and Sov. Math. Dokl., Vol.2, 1378-1381 (English translation).

Moreover,

$$\phi^* \equiv \sup_{r(0)\in R^2} \phi^+[r(t)] = \sup\sqrt{\tilde{\rho}_1 + \tilde{\rho}_2}\,, \tag{9}$$

where the supremum in the right-hand side of (9) is taken among all the solutions $\tilde{\rho}_1 \geq 0$*,* $\tilde{\rho}_2 \geq 0$ *of the algebraic system:*

$$\tilde{\rho}_s\left(\sum_{k_1+k_2=l} G_s^{(k_1,k_2)}\tilde{\rho}_1^{\,k_1}\tilde{\rho}_2^{\,k_2} + \frac{2}{N-1}\right) = 0,\ (\text{s=1,2},\ l = \left[\frac{N-1}{2}\right]). \tag{10}$$

For particular case $N=3$, the limit (8) is expressed as a function of the initial conditions and coefficients of (7) in subsection 3.4.3.

Theorem 3 reduces the problem of optimal stabilization to the construction of an admissible feedback control $u^* \in U_0$, that minimizes the value (9) defined by solutions of the algebraic system (10). A model example illustrating the above reduction is given in subsection 3.4.4.

Chapter 4 is devoted to the problem of stabilization with respect to a part of the variables. In section 4.1, the following non-autonomous system is considered:

$$\dot{y} = Y(t,y,z,u), \quad \dot{z} = Z(t,y,z,u), \tag{11}$$

$$(t \geq 0,\ y \in R^{n_1},\ z \in R^{n_2},\ u \in U \subseteq R^m,\ 0 \in U,\ Y(t,0,z,0) \equiv 0,\ Z(t,0,0,0) \equiv 0)$$

where $x = (y_1,\dots,y_{n_1},z_1,\dots,z_{n_2}) \in R^n$ is the state ($n=n_1+n_2$), u is the control. The right-hand side of (11) is supposed to be continuous on $D \times U$,

$$D = \{(t,y,z) : t \geq 0, |y| \leq H, z \in R^{n_2}\},\ (H=\text{const}>0).$$

We assume *z-extendability*[1] of the solutions, that is, for any measurable function $u(t)$: $[0,+\infty) \to U$, each solution $x(t) = (y(t), z(t))$ of (11) is defined for all $t \geq 0$ satisfying the inequality $|y(t)| \leq H$.

Let $V(t,x)$ be a function of class $C^1(D)$. We denote its time-derivative by virtue of (11) as $\dot{V}(t,x,u)$:

$$\dot{V}(t,x,u) = \frac{\partial V}{\partial t} + \sum_{j=1}^{n_1}\frac{\partial V}{\partial y_j}Y_j + \sum_{j=1}^{n_2}\frac{\partial V}{\partial z_j}Z_j.$$

In order to state precisely our results let us introduce two definitions.

Definition 2. A function $V(t,x) \in C^1(D)$ is said to be *control Lyapunov function with respect to* y if it satisfies the following conditions:

$$c_1(|y|) \leq V(t,x) \leq c_2(|y|), \quad c_1, c_2 \in K,$$

$$\forall (t,x) \in D \Rightarrow \inf_{u\in U} \dot{V}(t,x,u) \leq -\alpha(|y|), \quad \alpha \in K, \tag{12}$$

[1] Vorotnikov V.I. (1998): *Partial stability and control*. Birkhäuser, Boston.

where K is the Hahn functions class (i.e. K consists of all continuous, strictly increasing functions a: $[0,+\infty)\to[0,+\infty)$, $a(0)=0$).

Definition 3. A control Lyapunov function $V(t,x)\in C^1(D)$ with respect to y satisfies *the small control property* if, for each $\varepsilon>0$, $t_0\geq 0$, $x_0\in\{x: y=0\}$, there exists a neighborhood $B(t_0,x_0)$ of (t_0,x_0), such that

$$\forall (t,x)\in B(t_0,x_0)\cap D \Rightarrow \inf_{\substack{|u|<\varepsilon \\ u\in U}} \dot V(t,x,u)\leq -\alpha(|y|), \quad \alpha\in K. \tag{13}$$

The main result of section 4.1 is as follows.

Theorem 4. *Suppose that* U *is a compact set, and that* $V(t,x)$ *is a control Lyapunov function with respect to* y *for the system (11). Then there exists a (discontinuous) feedback law* $u: D\to U$, $u(t,0,z)=0$ *ensuring uniform asymptotic stability of the set* $M=\{x: y=0\}$*, provided that the solutions are defined in the sense of Filippov. Moreover, if* $D=[0,+\infty)\times R^n$*, and* $V(t,x)\to\infty$ *as* $|y|\to\infty$ *uniformly on* $(t,z)\in[0,+\infty)\times R^{n_2}$*, then the above feedback ensures global uniform asymptotic stability of* M.

Hereinafter, we consider a class of affine control systems:

$$\dot x = f_0(t,x)+\sum_{i=1}^{m} u_i f_i(t,x). \tag{14}$$

Theorem 5. *Let* $V(t,x)$ *be a control Lyapunov function with respect to* y *for (14), and let the above function satisfy the small control property,* $U=R^m$*. Then there exists a feedback control* $u(\cdot)\in C(D)$, $u(t,0,z)=0$*, that ensures uniform asymptotic stability of* $M=\{x: y=0\}$*. Such a feedback is given by the following expression:*

$$u_i(t,x)=\begin{cases} 0, & b(t,x)=0; \\ -\dfrac{b_i}{|b|}\left(a+\sqrt{a^2+|b|^4}\right), & b(t,x)\neq 0,\ 2\sqrt{a^2+|b|^4}\geq\alpha(|y|); \\ -\dfrac{b_i}{2|b|}\left(2a+\alpha(|y|)\right), & \text{otherwise}, \end{cases} \quad i=\overline{1,m}, \tag{15}$$

where $\alpha\in K$ *is an arbitrary Hahn function satisfying the inequalities (12), (13),*

$$a(t,x)=\frac{\partial V(t,x)}{\partial t}+\sum_{j=1}^{n}\frac{\partial V(t,x)}{\partial x_j}f_{oj}(t,x),\ b(t,x)=(b_1(t,x), b_2(t,x), \ldots, b_m(t,x)),$$

$$b_i(t,x)=\sum_{j=1}^{n}\frac{\partial V(t,x)}{\partial x_j}f_{ij}(t,x),\ i=\overline{1,m}. \tag{16}$$

If, moreover, $D=[0,+\infty)\times R^n$ *and* $V(t,x)\to\infty$ *as* $|y|\to\infty$ *uniformly on* $(t,z)\in[0,+\infty)\times R^{n_2}$*, then (15) ensures global uniform asymptotic stability of* M.

The result obtained extends Artstein's theorem[1] and Sontag's universal formula[2] for the case of stabilization with respect to a part of the variables.

In section 4.2, the problem of partial stabilization if considered for autonomous affine control systems. By applying the Barbashin – Krasovskii – LaSalle invariance principle, a pair of sufficient stabilizability conditions has been established.

Theorem 6. *Suppose that the system (14) is autonomous (i.e.* $f_i(t,x)$, $i=\overline{0,m}$ *don't depend on t). Let* $U=R^m$*, and let there be a function* $V(x)\in C^1(R^n)$ *satisfying the following conditions:*

1. $c_1(|y|)\le V(x)\le c_2(|y|)$, $\quad c_1,c_2\in K$;
2. *For each* x *of the closed domain*

$$D_H=\{x:\ |y|\le H,\ z\in R^{n_2}\},\ \ (H=\text{const}>0)$$

the time-derivative $\dot V(x,u)$ *by virtue of (14) has non-positive infimum:*

$$\inf_{u\in U}\dot V(x,u)\le 0;$$

3. *The system* $\dot x=f_0(x)$ *doesn't have any semitrajectory on* M_0 *defined for all* $t\ge 0$,

$$M_0=\left\{x\in D_H:\ \inf_{u\in U}\dot V(x,u)=0,\ \ y\ne 0\right\};$$

4. *For arbitrary* x_0: $b(x_0)=0$, $\varepsilon>0$*, there exists* $\delta(x_0,\varepsilon)>0$*:*

$$|x-x_0|<\delta\ \Rightarrow\ \inf_{|u|<\varepsilon}\dot V(x,u)\le 0;$$

5. *There exists* $\Delta_0>0$ *such that any solution* $x(t)=(y(t),z(t))$ *of the closed-loop system (14) with*

$$u_i(x)=-b_i\frac{a+\sqrt{a^2+|b|^4}}{|b|^2}\ \ \text{if } b(x)\ne 0;\quad u_i(x)=0\ \ \text{if } b(x)=0,\quad i=\overline{1,m},\quad (17)$$

is bounded for all $t\ge 0$*, provided that* $|y(0)|<\Delta_0$ *(functions* $a(x),b(x)$ *are given by (16)).*

Then the feedback (17) is continuous on D_H*, and the invariant set* $M=\{x\!: y=0\}$ *of the closed-loop system (14), (17) is asymptotically stable.*

Theorem 7. *Suppose that (14) is autonomous, and that* $U=R^m$*. Let* $V(x)\in C^1(R^n)$ *be a function satisfying the following conditions:*

1. $c_1(|y|)\le V(x)\le c_2(|y|)$, $\quad c_1,c_2\in K$;
2. *The equation*

$$a(x)+\sum_{i=1}^{m}u_i^0(x)\,b_i(x)=0$$

admits a continuous solution $u^0(x)$ *defined in* D_H*, such that the set*

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[1] Artstein Z. (1983): *Stabilization with relaxed controls*. Nonlinear Analysis, TMA, Vol.7, 1163-1173.

[2] Sontag E.D. (1989): *A universal construction of Artstein's theorem on nonlinear stabilization*. Systems and Control Letters, Vol.13, 117-123.

$$M_1=\{x\in D_H:\ b(x)=0,\ y\neq 0\}$$

doesn't have any positive semitrajectory of (14) with $u=u^0(x)$*;*

3. There exists a function $h\in C(D_H)$*,* $h(x)>0$*, such that each solution of the closed-loop system (14) with*

$$u_i(x)=u_i^0(x)-h(x)\,b_i(x),\qquad i=\overline{1,m},\tag{18}$$

is bounded with respect to z, provided that the initial values are small enough.

Then the feedback law (18) ensures asymptotic y-stability of the solution $x=0$ *of (14).*

The above results are applied in section 4.3 for deriving a feedback controller that solves the problem of attitude stabilization of a rigid body actuated by a pair of control torques.

In subsection 4.3.1, a problem on partial stabilization of a rigid satellite moving around its centroid by means of jet control torques is considered. The equations of motion are written in the form of Euler-Poisson:

$$\dot\omega_1=\frac{A_2-A_3}{A_1}\omega_2\omega_3+u_1;\ \ \dot\omega_2=\frac{A_3-A_1}{A_2}\omega_1\omega_3+u_2;\ \ \dot\omega_3=\frac{A_1-A_2}{A_3}\omega_1\omega_2;\tag{19}$$

$$\dot\nu_1=\omega_3\nu_2-\omega_2\nu_3\qquad (123),\tag{20}$$

where ω_i and ν_i denote the components of the angular velocity and the fixed attitude vector, respectively; A_i are the principal moments of inertia; u_1, u_2 denote the control torques actuated by jet engines. The system (19), (20) with $u_1=u_2=0$ admits the following solution:

$$\omega_1=\omega_2=\omega_3=0,\ \nu_1=\nu_2=0,\ \nu_3=1.\tag{21}$$

The solution (21) describes an equilibrium state when the third principal axis of inertia is collinear to the attitude vector ν. It should be emphasized, that the solution (21) cannot be asymptotically stabilized due to existence of an integral of (19), (20). In subsection 3.4.1, we consider the stabilization problem of the equilibrium (21) with respect to the following variables:

$$(\ \omega_1,\ \omega_2,\ \nu_1,\ \nu_2\ ).\tag{22}$$

The above variables correspond to the problem of single-axis stabilization of a body, at that the limit motions are rotations around the vector ν. By applying the results of section 4.2, we derive a feedback controller solving the partial stabilization problem:

$$u_1=\omega_2\omega_3-\frac{1}{A_1}\nu_2\nu_3-\left(\frac{|A_1-A_2|}{2A_2}|\omega_3|+\varepsilon A_1\right)\omega_1;$$

$$u_2=-\omega_1\omega_3+\frac{1}{A_2}\nu_1\nu_3-\left(\frac{|A_1-A_2|}{2A_1}|\omega_3|+\varepsilon A_2\right)\omega_2,\tag{23}$$

where ε is an arbitrary positive constant.

Thus, the feedback law (23) ensures asymptotic stability of the solution (21) of (19), (20) with respect to the variables (22). In addition, the equilibrium of the closed loop system (19), (20), (23) is proved to be stable in the sense of Lyapunov.

In subsection 4.3.2, we consider the case when the controls are implemented by a pair of rotating flywheels attached to a rigid body. The dynamical equations take the following form:

$$(A_1 - I_1)\dot{\omega}_1 = (A_2 - A_3)\omega_2\omega_3 + I_2\Omega_2\omega_3 - u_1;$$

$$(A_2 - I_2)\dot{\omega}_2 = (A_3 - A_1)\omega_1\omega_3 - I_1\Omega_1\omega_3 - u_2;$$

$$A_3\dot{\omega}_3 = (A_1 - A_2)\omega_1\omega_2 + I_1\Omega_1\omega_2 - I_2\Omega_2\omega_1;$$

$$I_1(\dot{\Omega}_1 + \dot{\omega}_1) = u_1; \quad I_2(\dot{\Omega}_2 + \dot{\omega}_2) = u_2, \tag{24}$$

where ω_i denote the angular velocity components of the body; Ω_1, Ω_2 – relative angular velocities of the flywheels; I_1, I_2 – inertia moments of the flywheels ($A_1 > I_1$, $A_2 > I_2$); u_1, u_2 denote the control torques applied to the flywheels. The system (24), (20) with $u_1=u_2=0$ admits a stationary motion:

$$\omega_1=\omega_2=\omega_3=0,\ \Omega_1=\text{const},\ \Omega_2=\text{const},\ \nu_1=\nu_2=0,\ \nu_3=1. \tag{25}$$

In subsection 4.3.2, we construct a feedback control for (24), (20), that stabilizes (25) with respect to the variables (22):

$$u_1 = \nu_2\nu_3 + (A_2\omega_2 + I_2\Omega_2)\omega_3 + \varepsilon\omega_1; \quad u_2 = -\nu_1\nu_3 - (A_1\omega_1 + I_1\Omega_1)\omega_3 + \varepsilon\omega_2, \tag{26}$$

where $\varepsilon > 0$ is an arbitrary constant.

The efficiency of the controllers proposed (23), (26) is illustrated by means of numerical simulation carried out in the thesis.

Chapter 5 is focused on mathematical modeling of a wind engine as a system of coupled rigid bodies. The model considered has five degrees of freedom and consists of three bodies: a pair of wind blades and a shaft having the fixed point (fig. 1). Each blade is attached by means of an elastic cylindrical joint, and its line of rotation is perpendicular to the longitudinal axis of the shaft. The above joints simulate resilience of the blades. The resilience of the shaft is taken into account by means of an elastic ball and socket joint placed at the fixed point. In section 5.1, we derive the motion equations for the model considered:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\varphi}_i} - \frac{\partial T}{\partial \varphi_i} = -\frac{\partial \Pi}{\partial \varphi_i}, \ i=1, 2;$$

$$\dot{L} + \omega \times L = M; \ \dot{\gamma} + \omega \times \gamma = 0, \tag{27}$$

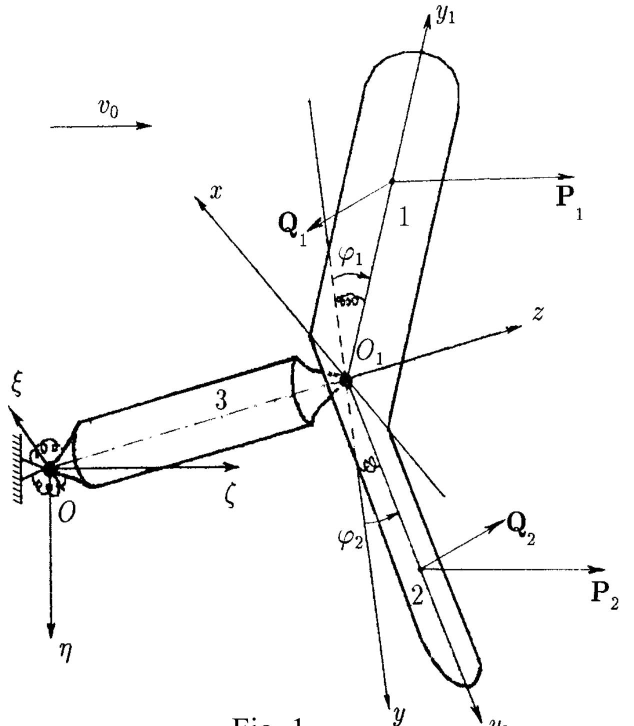

Fig. 1.

where φ_1, φ_2 denote the deviation angles of the first and the second blade, respectively; T is the kinetic energy; Π is the potential of the axial pressure and elasticity forces at the cylindrical joints; L is the angular momentum about the fixed point; $\omega = (\omega_1, \omega_2, \omega_3)$ is the angular velocity of the shaft; M is the moment of force about the fixed point; the vector $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ is fixed and collinear to the air flow. In the scalar form, the system (27) consists of eight nonlinear ordinary differential equations with regard to $\varphi_1, \varphi_2, \omega_1, \omega_2, \omega_3, \gamma_1, \gamma_2, \gamma_3$.

In section 5.2, we construct the system of linear approximation for (27) in a neighborhood of the following stationary motion:

$$\varphi_1 = \varphi_2 = \varphi_0 = \text{const}, \;\; \omega_1 = \omega_2 = 0, \;\; \omega_3 = \omega_0 = \text{const} \neq 0, \; \gamma_1 = \gamma_2 = 0, \; \gamma_3 = 1. \qquad (28)$$

The solution (28) corresponds to on-speed conditions of the wind engine. The characteristic polynomial of the linearized system is factorized as the product of polynomials of the first, third, and sixth degrees.

In section 5.3, we investigate stability conditions of the solution (28). Domains of stability in the set of basic mechanical parameters are constructed numerically.

An individual case without damping is considered in subsection 5.3.1. In that case, the necessary stability conditions are analyzed for sufficiently small values of the parameters φ_0 and σ/κ, where $\kappa>0$, $\sigma>0$ denote the stiffness coefficients of the cylindrical and ball and socket joints, respectively. We show that the stability condition is reduced to the following inequality:

$$\frac{\omega_0^{\,2}}{\kappa} > \frac{2A_1 - J_{xx} + J_{zz} - 2ml^2}{A_1(J_{xx} - J_{zz} + 2ml^2)},$$

where A_1 is the inertia moment of a blade with respect to the axis of cylindrical joint; J_{xx}, J_{zz} are the inertia moments of the shaft; m is the mass of a blade; l is the length of the shaft.

The Lienard-Chipart sufficient conditions of stability are investigated in subsection 5.3.2 for the case with damping. Two extreme cases corresponding to large values of the stiffness coefficients are considered.

Case 1: the stiffness coefficient $\kappa>0$ *is bounded, while* $1/\sigma$ *is a small parameter.* We have proved asymptotic stability of the solution (28) in that case.

Case 2: the coefficient σ *is bounded, while* $1/\kappa$ *is a small parameter.* In the presence of an additional constraint, we get the following condition for asymptotic stability of (28):

$$P < \frac{\tau_1{}^2 \left\{ q\delta^* + \sqrt{(q\delta^*)^2 + ((A_1 - A_2)^2 - q^2)\,(\delta^* + (\nu\tau / \tau_1)^2)} \right\}}{\nu^2 r\,((A_1 - A_2)^2 - q^2)},$$

$$q = A_1 + A_2 + J_{zz} - J_{xx} - 2ml^2, \quad \delta^* = (\sigma + 2Pl)/(Pr),$$

where P is the force of frontal pressure; τ_1, τ are positive damping coefficients; A_2 is the inertia moment of a blade; ν is the ratio of the turning moment and the moment of frontal pressure; r is the distance between the line of rotation of the shaft and the aerodynamic center of a blade.

In Appendix A, we give the coefficients of the characteristic polynomial for the problem considered in Chapter 5.

Conclusions

A question of sufficient stabilizability conditions is investigated for nonlinear control system governed by ordinary differential equations. For the problem of partial stabilization, we have proposed a controller design scheme based on the concept of control Lyapunov functions. In addition, problems of stability and stabilization have been solved for some systems of coupled rigid bodies. The main results of thesis are as follows:

1. It has been proved that, for any locally controllable system, there exists a (discontinuous) feedback law ensuring Lyapunov stability of the equilibrium. At that, the closed-loop solutions are defined in the sense of Filippov. For affine control systems, it has been shown that the set of discontinuities of the above feedback is contained in a proper submanifold of the state space.
2. For the critical case of two pairs of purely imaginary roots, asymptotic estimates of the solutions have been obtained. The above estimates are shown to be applicable for solving the problem of optimal stabilization.
3. Sufficient conditions for non-autonomous systems to be stabilizable have been derived by means of the control Lyapunov functions with respect to a part of the variables. For affine systems, a constructive approach for feedback design has been developed. This result extends Artstein's theorem for the case of partial stabilization.
4. Sufficient conditions for partial stabilizability of nonlinear autonomous systems have been obtained, provided that there exists a Lyapunov function having non-positive lower bound of time-derivatives. The problem of partial stabilization of a rigid body subjected to a pair of control torques has been solved by means of the above technique. Two cases have been considered. In the first case the control is implemented by jet engines, while in the second one the satellite is controlled by means of moving masses (flywheels).
5. A mathematical model of a wind engine unit has been proposed in the form of a system of coupled rigid bodies. Stability conditions of a stationary motion have been obtained by means of the linearization procedure. For the case of small blade deviation and large stiffness coefficients of the shaft, we have got a stability condition that establishes a relation of the angular velocity with the stiffness and inertial parameters. Under an additional constraint, we have obtained a sufficient stability condition imposing a restriction on admissible forces of frontal pressure. It has been established that uniform rotation of the model is stable when the stiffness coefficient of the shaft is large enough.